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Analysis of clamped unsymmetric cross-ply rectangular plates by superposition of simple exact double Fourier series solutions

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Abstract

Based on the idea developed in an earlier work on orthotropic rectangular plates [Compos. Struct. 63 (2004) 63], it is shown here that exact double Fourier series solutions can be easily obtained for simply supported rectangular unsymmetrically laminated cross-ply plates subjected to various edge forces and moments, and that these can be superposed to yield a simple solution for the title problem considered difficult hitherto. Numerical results, based on the summation of the infinite series without truncation using MATLAB, are presented for a set of symmetric and antisymmetric cross-ply laminates under uniform and central concentrated loads.

1. Introduction

Clamped rectangular plates under arbitrary loads are not easily amenable to rigorous analysis, and are hence usually analyzed using approximate methods like the Ritz method or finite element method. Of the available rigorous solutions for clamped isotropic plates, Timoshenko's work [2] based on the superposition of Levy-type solutions for simply supported plates under uniform or central concentrated loading and various edge moments, is the simplest and most well-known. Other approaches are based on superposition of more complicated solutions [3] or on the use of the mathematically complex boundary-discontinuous Fourier series originally put forth by Green [4]. All these solutions are in the form of infinite series and have to be termed exact in the sense that the governing differential equation and boundary conditions can be satisfied to any desired degree of accuracy by taking more and more terms.

For laminated orthotropic plates, the direct extension of Timoshenko's superposition approach is impractical because the development of the component Levy-type solutions, now required for edge forces as well as moments, becomes very tedious; even for a single-layer orthotropic plate, the nature of the solution varies depending on the orthotropic properties (see Ref. [2], p. 373), while for an unsymmetric laminate one has to solve three coupled ordinary differential equations.

The objective of the present work is to demonstrate that an application of the principle of virtual work enables one to obtain double Fourier series solutions for a simply supported arbitrarily laminated cross-ply plate subjected to various edge forces and moments, and that these infinite series solutions are the exact equivalents of the corresponding closed-form Levy-type solutions. Carrying out the series summation *without truncation* using *MATLAB*, and by appropriate superposition, one can analyze a fully clamped plate subjected to any loading using this approach and obtain results of any desired accuracy. The simplicity of this approach has been demonstrated earlier [1] with reference to single-layer or symmetrically laminated orthotropic plates with any combination of simply supported and clamped edges.

2. Methodology

For clarity of presentation, a plate subjected to uniform load q_0 or a central concentrated load P is considered first. Extending Timoshenko's approach to an unsymmetric cross-ply clamped plate under such loading, one can consider it as a simply supported plate subjected to (i) the applied load, (ii) bending moments

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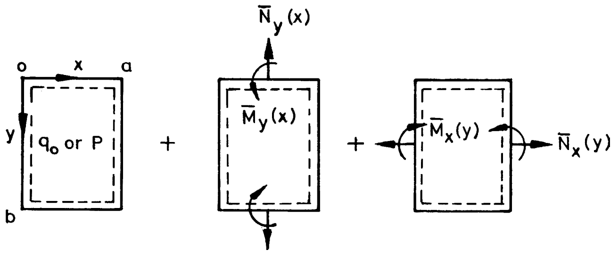

Fig. 1. Superposition of different load cases.

and in-plane normal forces applied along a pair of opposite edges, and (iii) similar edge moments and forces acting on the other pair of opposite edges (Fig. 1), with the following simple support conditions:

$$
\begin{aligned}
&\text{at } x = 0, a: \quad w = M_x = N_x = v_0 = 0\\
&\text{at } y = 0, b: \quad w = M_y = N_y = u_0 = 0
\end{aligned}
\tag{1}
$$

u_0, v_0 and w are the x, y, z displacements, respectively of a point on the mid-plane, and N_x, M_x, etc. are related to the mid-plane strains and curvatures in terms of the well-known stiffness coefficients A_{ij}, B_{ij} and D_{ij} [5] as

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix}
\tag{2}
$$

The next step is to obtain the exact solutions for the simply supported plate loaded as in Fig. 1, and this is carried out by an application of the principle of virtual work as given by

$$
\begin{aligned}
&\int\int [N_x \delta u_{0,x} + N_y \delta v_{0,y} + N_{xy}(\delta u_{0,y} + \delta v_{0,x})\\
&\qquad - M_x \delta w_{,xx} - M_y \delta w_{,yy} - 2M_{xy}\delta w_{,xy}]\, \mathrm{d}x\, \mathrm{d}y\\
&\quad = \left[P\delta w|_{(a/2,b/2)} \quad \text{or} \quad q_0 \int\int \delta w\, \mathrm{d}x\, \mathrm{d}y \right]\\
&\qquad - 2\int_0^b \overline{N}_x \delta u_0|_{x=0}\, \mathrm{d}y + 2\int_0^b \overline{M}_x \delta w_{,x}|_{x=0}\, \mathrm{d}y\\
&\qquad - 2\int_0^a \overline{N}_y \delta v_0|_{y=0}\, \mathrm{d}x + 2\int_0^a \overline{M}_y \delta w_{,y}|_{y=0}\, \mathrm{d}x
\end{aligned}
\tag{3}
$$

Seeking Fourier series solutions satisfying the simple support conditions (Eq. (1)) as

$$
\begin{aligned}
u_0 &= \sum_m \sum_n U_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\\
v_0 &= \sum_m \sum_n V_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\\
w &= \sum_m \sum_n W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
\end{aligned}
\tag{4}
$$

where m and n take only odd values in view of the symmetry of deformation, and taking the undetermined edge loads as

$$
\begin{aligned}
(\overline{N}_x, \overline{M}_x) &= \sum_{n=1,3,\ldots} (N_{xn}, M_{xn}) \sin\frac{n\pi y}{b}\\
(\overline{N}_y, \overline{M}_y) &= \sum_{m=1,3,\ldots} (N_{ym}, M_{ym}) \sin\frac{m\pi x}{a}
\end{aligned}
\tag{5}
$$

one can substitute them in Eq. (3) along with Eq. (2), and obtain the following equations in view of the orthogonality of the sine and cosine functions.

$$
\begin{aligned}
&U_{mn}(A_{11}\alpha^2 + A_{66}\beta^2) + V_{mn}(A_{12} + A_{66})\alpha\beta\\
&\quad - W_{mn}(B_{11}\alpha^3 + (B_{12} + 2B_{66})\alpha\beta^2) = -4N_{xn}/a\\
&U_{mn}(A_{12} + A_{66})\alpha\beta + V_{mn}(A_{22}\beta^2 + A_{66}\alpha^2)\\
&\quad - W_{mn}(B_{22}\beta^3 + (B_{12} + 2B_{66})\alpha^2\beta) = -4N_{ym}/b\\
&- U_{mn}(B_{11}\alpha^3 + (B_{12} + 2B_{66})\alpha\beta^2)\\
&\quad - V_{mn}(B_{22}\beta^3 + (B_{12} + 2B_{66})\alpha^2\beta)\\
&\quad + W_{mn}(D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4)\\
&\quad = \left(\frac{4P}{ab} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \ \text{or} \ \frac{16q_0}{mn\pi^2} \right) + \frac{4m\pi M_{xn}}{a^2} + \frac{4n\pi M_{ym}}{b^2}
\end{aligned}
\tag{6}
$$

for each m, n, where $\alpha = m\pi/a$ and $\beta = n\pi/b$.

These equations readily yield U_{mn}, etc. in terms of the unknowns M_{xn}, etc. The final solution is of the form

$$
\begin{aligned}
u_0 &= (P \text{ or } q_0) \sum_m \sum_n U_{1mn} \cos\alpha x \sin\beta y\\
&\quad + \sum_m N_{ym} \cos\alpha x \sum_{n=1}^{\infty} U_{2mn} \sin\beta y\\
&\quad + \sum_m M_{ym} \cos\alpha x \sum_{n=1}^{\infty} U_{3mn} \sin\beta y\\
&\quad + \sum_n N_{xn} \sin\beta y \sum_{m=1}^{\infty} U_{4mn} \cos\alpha x\\
&\quad + \sum_n M_{xn} \sin\beta y \sum_{m=1}^{\infty} U_{5mn} \cos\alpha x
\end{aligned}
\tag{7a}
$$

$$
\begin{aligned}
v_0 &= (P \text{ or } q_0) \sum_m \sum_n V_{1mn} \sin\alpha x \cos\beta y\\
&\quad + \sum_m N_{ym} \sin\alpha x \sum_{n=1}^{\infty} V_{2mn} \cos\beta y\\
&\quad + \sum_m M_{ym} \sin\alpha x \sum_{n=1}^{\infty} V_{3mn} \cos\beta y\\
&\quad + \sum_n N_{xn} \cos\beta y \sum_{m=1}^{\infty} V_{4mn} \sin\alpha x\\
&\quad + \sum_n M_{xn} \cos\beta y \sum_{m=1}^{\infty} V_{5mn} \sin\alpha x
\end{aligned}
\tag{7b}
$$

$$\begin{aligned}w = (P \text{ or } q_0)\sum_m\sum_n W_{1mn}\sin\alpha x\sin\beta y \\ + \sum_m N_{ym}\sin\alpha x\sum_{n=1}^{\infty} W_{2mn}\sin\beta y \\ + \sum_m M_{ym}\sin\alpha x\sum_{n=1}^{\infty} W_{3mn}\sin\beta y \\ + \sum_n N_{xn}\sin\beta y\sum_{m=1}^{\infty} W_{4mn}\sin\alpha x \\ + \sum_n M_{xn}\sin\beta y\sum_{m=1}^{\infty} W_{5mn}\sin\alpha x\end{aligned} \tag{7c}$$

where U_{1mn}, V_{1mn}, W_{1mn}, etc. are functions of m, n and the stiffness coefficients A_{ij}, B_{ij} and D_{ij}.

A careful look at Eqs. (7a)–(7c) indicates that the first terms correspond to a Navier-type solution for the simply supported plate under the uniform or concentrated load, the second terms to the solution for the plate under $\overline{N}_y$, the third terms to that for $\overline{M}_y$, and so on. Further, the inner infinite summation in the second term of any of these equations represents the *exact solution* corresponding to any particular harmonic $N_{ym}\sin\alpha x$, and so on. This can best be understood by considering the case of a homogeneous isotropic plate under uniform load. For this case, due to the absence of bending–stretching coupling, one gets only one equation in place of Eq. (6), as given by

$$W_{mn}D(\alpha^2+\beta^2)^2 = \frac{16q_0}{mn\pi^2} + \frac{4m\pi M_{xn}}{a^2} + \frac{4n\pi M_{ym}}{b^2} \tag{8}$$

and hence

$$\begin{aligned}w = q_0\sum_m\sum_n \frac{16}{mn\pi^2 D(\alpha^2+\beta^2)^2}\sin\alpha x\sin\beta y \\ + \sum_m M_{ym}\sin\alpha x\sum_{n=1}^{\infty}\frac{4n\pi}{b^2D(\alpha^2+\beta^2)^2}\sin\beta y \\ + \sum_n M_{xn}\sin\beta y\sum_{m=1}^{\infty}\frac{4m\pi}{a^2D(\alpha^2+\beta^2)^2}\sin\alpha x\end{aligned} \tag{9}$$

where D is the flexural rigidity of the plate.

The first term in the above equation is clearly the Navier-type solution for the uniformly loaded plate [2]. To understand the significance of the second term, one has to look at the exact Levy-type solution for the case of symmetrical bending due to harmonic moments $M_{ym}\sin\alpha x$ applied at the edges $y = 0$ and $y = b$. This is given by [2]

$$\begin{aligned}w = \left(\frac{a^2M_{ym}\sin\alpha x}{2\pi^2Dm^2\cosh p_m}\right)[p_m\tanh p_m\cosh\alpha(y-b/2) \\ - \alpha y\sinh\alpha(y-b/2)]\end{aligned} \tag{10}$$

where $p_m = m\pi b/2a$.

A half-sine series expansion of the above solution in the interval $(0 \leqslant y \leqslant b)$ would give the second term of Eq. (9); hence, it is clear that the principle of virtual work yields the series equivalent of the exact closed-form Levy-type solution for harmonically varying edge loading. Thus, having established the legitimacy of the solution given by Eq. (7), the last step is to determine the unknown force and moment coefficients N_{xn}, M_{xn}, N_{ym} and M_{ym}. These are obtained by imposing the rigidly clamped boundary conditions for the net displacements and slopes as

$$\begin{aligned}&\text{at } x = 0, a: \quad w = w_{,x} = u_0 = v_0 = 0 \\ &\text{at } y = 0, b: \quad w = w_{,y} = u_0 = v_0 = 0\end{aligned} \tag{11}$$

It should be noted that two of the above conditions per edge have already been enforced (Eq. (1)), and symmetry of deformation has been accounted for; hence, the unimposed conditions lead to four infinite sets of equations in terms of the force and moment coefficients. For example, corresponding to zero u_0 along the edge $x = 0$, one gets

$$\begin{aligned}(P \text{ or } q_0)\sum_m^{\infty} U_{1mn} + \sum_m N_{ym}U_{2mn} + \sum_m M_{ym}U_{3mn} \\ + N_{xn}\sum_{m=1}^{\infty} U_{4mn} + M_{xn}\sum_{m=1}^{\infty} U_{5mn} = 0 \quad \text{for } n = 1, 3, \ldots, \infty\end{aligned} \tag{12}$$

By considering a finite number of force and moment coefficients depending on the degree of accuracy desired, one gets four finite sets of equations which can be solved. It is very important to note that the infinite sums occurring in any of these equations have to be summed *without truncation.*

At this stage, it is appropriate to point out that the more general case of arbitrary loading can be handled in a similar manner, except that even m and n terms also appear in the equations, and one has to enforce the clamped edge conditions at each edge separately.

3. Validation of the present approach

In order to validate the present series approach, the case of a homogeneous isotropic square plate under uniform/central concentrated load is considered; for this problem, the maximum deflection and the bending moments, based on superposition of closed-form Levy-type solutions, are available [2].

Analyzing the case of uniform load by the present approach, one has $M_{xn} = M_{yn}$ due to symmetry of deformation, and obtains the following single set of equations by enforcing the slope to be zero at $x = 0$.

$$\begin{aligned}q_0\sum_{m=1}^{\infty}\frac{16\alpha}{mn\pi^2D(\alpha^2+\beta^2)^2} + \sum_m \frac{4n\pi\alpha M_{ym}}{a^2D(\alpha^2+\beta^2)^2} \\ + M_{yn}\sum_{m=1}^{\infty}\frac{4m\pi\alpha}{a^2D(\alpha^2+\beta^2)^2} = 0 \quad \text{for } n = 1, 3, \ldots\end{aligned} \tag{13}$$

Table 1
Validation study—the case of an isotropic square plate

Item	$m = 1, 3, \ldots, 31$	$m = 1, 3, \ldots, 51$	$m = 1, 3, \ldots, 101$	$m = 1, 3, \ldots, \infty$	Ref. [2]
Uniform load					
$(D/q_0a^4)w$ at center	0.00120	0.00123	0.00125	0.00126	0.00126
M_x/q_0a^2 at $(0, a/2)$	−0.0529	−0.0523	−0.0519	−0.0513	−0.0513
M_x/q_0a^2 at center	0.0237	0.0234	0.0223	0.0231	0.0231
Central conc. load					
$(D/Pa^2)w$ at center	0.00547	0.00552	0.00557	0.00560	0.00560
M_x/P at $(0, a/2)$	−0.1305	−0.1289	−0.1274	−0.1257	−0.1257

Note: Results obtained using moment coefficients up to M_{x13} and M_{y13} and Poisson's ratio = 0.3.

The infinite sums occurring in the above equation as well as those in Eq. (9) can be evaluated using *MATLAB* by specifying the upper limit as infinity. (This feature is also available in other mathematical software such as *MATHEMATICA*.) Results using such untruncated summation as well as those based on various truncated approximations are compared with the benchmark results of Ref. [2] in Table 1; such a comparison for the case of the central concentrated load is also included therein.

From these results, it is clear that the present methodology reproduces the results of the exact Levy-type approach when the infinite summations are carried out using *MATLAB*; further, it is also clear that truncated evaluation leads to significant error, especially for the moments.

4. Numerical results for cross-ply plates

Results are obtained for square plates, of total thickness h, with the following lay-ups—(0°), (0°/90°), (0°/90°/0°) and (0°/90°/0°/90°). The material properties are taken to be $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$ and $\upsilon_{LT} = 0.25$. The number of force and moment coefficients N_{xn}, M_{xn}, etc. to be taken is decided based on a convergence study. The typical convergence pattern is as shown in Table 2; it is clear that all the results converge rapidly, with deflections converging much earlier compared to the moments and forces as can be expected.

Converged results for the four lay-ups considered are presented in Tables 3 and 4. For the antisymmetric lay-ups, the moments M_x and M_y are equal at the center and at corresponding mid-edges, while the mid-edge normal forces N_x and N_y are equal but of opposite sign, as can be easily visualized. Of all the lay-ups, the deflections are largest for the (0°/90°) plate because of the high bending–stretching coupling. For uniform load, irrespective

Table 2
Convergence of the results for a (0°/90°) plate under a central concentrated load

m, n up to[a]	$(E_Th^3/Pa^2)w$ at center	M_y/P at $(a/2, 0)$	N_xa/P at $(0, a/2)$
7	0.01783	−0.1307	−0.01086
11	0.01783	−0.1304	−0.01078
15	0.01783	−0.1303	−0.01071
25	0.01783	−0.1303	−0.01058
35	0.01783	−0.1303	−0.01062
45	0.01783	−0.1303	−0.01060

[a] See Eq. (5).

Table 3
Results for square laminates under uniform load

Lay-up	$(E_Th^3/q_0a^4)w$ at center	M_x/q_0a^2 at center	M_y/q_0a^2 at center	M_x/q_0a^2 at $(0, a/2)$	M_y/q_0a^2 at $(a/2, 0)$	N_x/q_0a at $(0, a/2)$
(0°)	0.001308	0.04389	0.0006898	−0.08703	−0.01121	–
(0°/90°)	0.003954	0.02387	0.02387	−0.05534	−0.05534	−0.00177
(0°/90°/0°)	0.001371	0.04441	0.001838	−0.08809	−0.01569	–
(0°/90°/0°/90°)	0.001761	0.02469	0.02469	−0.05649	−0.05649	−0.000398

Note: $N_x(0, a/2) = -N_y(a/2, 0)$ for the antisymmetric laminates.

Table 4
Results for square laminates under a central concentrated load

Lay-up	$(E_Th^3/Pa^2)w$ at center	M_x/P at $(0, a/2)$	M_y/P at $(a/2, 0)$	N_xa/P at $(0, a/2)$
(0°)	0.009167	−0.3957	0.002090	–
(0°/90°)	0.01783	−0.1303	−0.1303	−0.0106
(0°/90°/0°)	0.008175	−0.3361	−0.002476	–
(0°/90°/0°/90°)	0.007971	−0.1334	−0.1334	−0.00235

Note: $N_x(0, a/2) = -N_y(a/2, 0)$ for the antisymmetric laminates.

of the lay-up, the mid-edge moments are always larger than the corresponding central moments.

These tabulated results will be useful for judging the accuracy of various approximate methods which are frequently employed for clamped plate problems.

5. Conclusion

The problem of a general cross-ply rectangular plate clamped at all four edges and subjected to arbitrary loading, considered difficult hitherto, has been shown here to admit of a simple solution based on a direct extension of Timoshenko's superposition approach; the need of tedious analytical manipulations involved in deriving the component Levy-type solutions is obviated by the ingenious use of their equivalent Fourier series which are summed without truncation. The methodology enables one to analyze clamped laminates easily without resorting to approximate methods and is valuable in view of the fact that results for such plates, even for commonly encountered loads, cannot be tabulated in handbooks because of their dependence on the several orthotropic stiffness coefficients.

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